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Non-isometric between the Topological spaces $(R^n, d_2), (R^n, d_\infty)$ for every dimension, $n \geq 2$

By: Wafiq Hibi

Assistant Professor

Wafiq. hibi@gmail.com, Wafiq. hibi@sakhnin.ac.il

Head of the Mathematics Department in Sakhnin College.

The College of Sakhnin

Academic College for Teacher Education

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Abstract:

We definition A metric space X is a topological space in which the topology is given by a metric, or distance function, d, which is a non-negative, real valued mapping of $X \times X$ with the following properties:

For all $x, y, z \in X$

(1) $d(x, y) = 0$ iff $x = y$,

(2) $d(x,y) = d(y,x)$, and

(3) $d(x,z) \leq d(x,y) + d(y,z)$.

We shall denote by R the field of real numbers. Then we shall use the Cartesian product

$R^n = R \times R \times . . . \times R$ of ordered n-tuples of real numbers (n factors).

Typical notation for $x \in R^n$ will be $x = (x_1, \ldots, x_n)$.

It is known that:

a. The n-dimensional Euclidean space R^n with the "usual distance", this is

Sometimes called the 2-metric d_2.

$$d_2\,(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\,.$$

b. The n-dimensional Euclidean space R^n with the "taxi cab" metric, this is often

called the 1-metric d_1.

$$d_1\,(x, y) = \sum_{i=1}^{n}|x_i - y_i|.$$

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c. The n-dimensional Euclidean space R^n with "supremum" or "maximum" metric, It is often called the infinity metric d_∞.

$d_\infty(x, y) = max_{1 \le i \le n} \{|x_i - y_i|\}$.

All of them are metric spaces [1, 2], and denote this metric space $(R^n, d_2), (R^n, d_1), (R^n, d_\infty)$ respectively.

Let f be a function from a metric space (X, d) on a metric space (Y, ρ).

We say that f is an isometry if $d(x, y) = \rho\big(f(x), f(y)\big)$, for any $x, y \epsilon X$.

It is known that the two spaces $(R^2, d_1), (R^2, d_\infty)$ are isometric. It is known, too, that the space (R^2, d_2) is not isometric for both of them [3].

The proof of that (the space $(R^2, d_2)$is not isometric for both $(R^2, d_1), (R^2, d_\infty)$) is based on that (R^2, d_2) is not isometric for (R^2, d_∞) , and that concludes that (R^2, d_2) is not isometric for (R^2, d_1), too [3].

Here, we extend this conclusion by showing that the two spaces $(R^n, d_2), (R^n, d_\infty)$ are not isometric for every dimension, $n \geq 2$.

Keywords:"Metric spaces", "usual distance", "maximum metric", "taxi cab metric", and "isometric between metric spaces".

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تلخيص البحث:

تعريف 1:

نعرّف بالرياضيات فراغ قياسي X، على أنه فراغ طوبولوجي بحيث أن الطوبولوجيا المعطاة عليه هي قياس أي دالة بُعد، معرفة كالتالي: $d: X \times X \to [0, \infty)$

وبحيث يتحقق لكل، $x, y, z \in X$:

1) $x = y$ اذا وفقط اذا $d(x, y) = 0$.
2) $d(x,y) = d(y,x)$.
3) $d(x,z) \leq d(x,y) + d(y,z)$.

وعندها يقال للزوج المرتب (X, d) المكوّن من الفراغ ودالة البعد المعرّفة عليه على أنه فضاء قياسيّ.

تعريف 2:

الفراغ النونيّ او الفراغ الإقليدي من البعد العام والذي يرمز له R^n، n هو عدد صحيح موجب، يعرّف كالتالي:

$$R^n = \{(x_1, \dots, x_n): x_i \in R, 1 \leq i \leq n\}$$

حقائق 1:

- معروف أن الفراغ النونيّ صاحب البعد العام R^n مع دالة البعد العادية d_2 ، على أنه فضاء قياسي،[3]، ويرمز له (R^n, d_2). الدالة d_2 تسمى دالة البعد العادية أو دالة البعد الإقليدية وهي معرفة:

$$d_2\ (x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

- كذلك الفراغ النوني صاحب البعد العام R^n مع دالة البعد لمجموع القيم المطلقة d_1، هو أيضًا فضاء قياسي،[3]، ويرمز له (R^n, d_1). الدالة d_1 تسمى دالة البعد لمجموع القيم المطلقة وهي معرفة:

$$d_1\ (x, y) = \sum_{i=1}^{n}|x_i - y_i|$$

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- أيضًا الفراغ النوني صاحب البعد العام R^n مع دالة البعد العظمى d_∞ , والذي يرمز له (R^n, d_∞)، كذلك هو فضاء قياسي،[3]. الدالة d_∞ تسمى دالة البعد العظمى وهي معرفة:

$$d_\infty\ (x, y) = max_{1\leq i\leq n}\ \{|x_i - y_i|\}.$$

تعريف 3:

نقول أن فضاءين قياسيين (X, d) و (Y, ρ) متساويان فيما بينهما بالقياس إذا وجدت دالة f،

$$f: (X, d) \to (y, \rho)$$

تحقق:

1) الدالة f من الفضاء القياسي (X, d) على الفضاء القياسي (Y, ρ).
2) يتحقق لكل $x, y \in X$: $d(x, y) = \rho\big(f(x), f(y)\big)$.

يذكر ان الدالة تسمى بهذه الحالة دالة المقياس بين الفضاءين (X, d) و (Y, ρ).

حقائق 2:

- معروف أن الفراغ النونيّ ثنائي الابعاد R^2 مع دالة البعد لمجموع القيم المطلقة d_1، أي الفضاء القياسي (R^2, d_1) والفراغ النوني ثنائي الابعاد R^2 مع دالة البعد العظمى أي الفضاء القياسي (R^2, d_∞)، هما فضاوان متساويا القياس، [1,4].
- معروف أيضًا أن الفراغ النونيّ ثنائي الأبعاد R^2 مع دالة البعد العادية d_2، أي الفضاء القياسي (R^2, d_2) غير متساوي القياس مع كلا الفضاءين القياسيين (R^2, d_∞) و (R^2, d_1)،[2].

لبرهان الحقيقة الثانية يكفي برهنة أن الفضاءين (R^2, d_∞) و (R^2, d_2) غير متساويين بالقياس وذلك اعتمادًا على الحقيقة الأولى حيث ان الفضاءين (R^2, d_∞) و (R^2, d_1) هما نعم متساويا القياس.

في هذا المقال سنوسّع هذه النتيجة ونبرهن أن الفضاءين: (R^n, d_∞) و (R^n, d_2) غير متساويين في القياس لكل بعد $n \geq 2$.

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اجمال:

ملخص هذا المقال أنه لدينا نتيجة رئيسية هي البرهان الرياضي للنظرية التالية:

نظرية:

أن الفراغ النونيّ صاحب البعد العام R^n مع دالة البعد العظمى d_∞ , أي الفضاء القياسي (R^n, d_∞)،

والفراغ النونيّ صاحب البعد العام R^n مع دالة البعد العادية d_2 ، اي فضاء قياسي (R^n, d_2)،

غير متساويين في القياس لكل بعد $n \geq 2$.

كلمات مفتاحية: "فضاء قياسي"، "البعد الإقليدي العادي"، "البعد لمجموع القيم المطلقة"، "دالة البعد ا لعظمى"، "فضاءات متساوية القياس".

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Our main result is the following theorem:

Theorem:

The two spaces $(R^n, d_2), (R^n, d_\infty)$ are not isometric for every dimension ,$n \geq 2$.

Proof:

We suppose by contradiction that there is an isometry

$$f\colon (R^n, d_\infty) \to (R^n, d_2).$$

Assume without loss of generality that f maps the origin to the origin. That means,

$$f(0,0 \ldots\ldots,0) = (0,0 \ldots\ldots,0),$$

Because if not, and:

$$f(0,0 \ldots\ldots,0) = (a_1, \ldots\ldots, a_n)$$

When $a_i \in R$ for all $1 \leq i \leq n$, and not all of them are zeros.

By looking at this function:

$$g\colon (R^n, d_\infty) \to (R^n, d_\infty)$$

When $g(x_1, \ldots\ldots, x_n) = (x_1 - a_1, \ldots\ldots, x_n - a_n)$ for all $x = (x_{i=1}^n) \in R^n$.

It is clear that g is isometry because it is a sliding function, and the composing function:

$$g \circ f :\ (R^n, d_\infty) \to (R^n, d_2).$$

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Is isometry, depending on the state that the composition of an isometry function is isometry as well.

By that:

$$(g \circ f\ )(0, \dots ,0) = g\big(f(0, \dots ,0)\big) = g(a_1, \dots , a_n)$$

$$= (a_1 - a_1, \dots , a_n - a_n)$$

$$= (0, \dots ,0)$$

That means that $g \circ f$ is an isometry and maps the origin to the origin.

From here, it is possible to assume that there is an isometry function

$$\varphi: (R^n, d_\infty) \to (R^n, d_2)$$

Moreover, it maps origin to origin:

Looking at the following points into (R^n, d_∞):

$$0: (0, \dots ,0)$$

$$x: (1, -1, \dots ,0)$$

$$y: (-1, -1,0 \dots . ,0)$$

$$z: (0\ ,1\ ,0\ , \dots ,0)$$

It is clear that:

$$d_\infty(x, 0) = d_\infty(y, 0) = d_\infty(z, 0) = 1.$$

That means that $x, y\ and\ z$ are three points at a unit circle that its center is the origin.

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Moreover, it is clear, as well. That their (the three points x, y and z) mutual distance is 2 from each other:

$$d_\infty(x,y) = d_\infty(y,z) = d_\infty(x,z) = 2.$$

Now, by looking at the following points at (R^n, d_2):

$$\varphi(0), \varphi(x), \varphi(y), \varphi(z)$$

It is clear that $\varphi(0) = 0$.

Since that φ is an isometry, and specifically, preserves distances,

Causing that:

$$d_2\big(0, \varphi(x)\big) = d_2\big(0, \varphi(y)\big) = d_2\big(0, \varphi(z)\big) = 1$$

At the space (R^n, d_2).

Which means that the three points $\varphi(x), \varphi(y), \varphi(z)$ are three points at a unit circle that its center is in the origin.

Furthermore, the mutual distances between each other are 2, because of the fact that ψ is an isometry.

$$d_2\big(\varphi(x), \varphi(y)\big) = d_2\big(\varphi(y), \varphi(z)\big) = d_2\big(\varphi(x), \varphi(z)\big) = 2.$$

Which is contradiction, this state cannot happen at the space (R^n, d_2).

This completes the proof of the theorem.

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Main result:

Our main result is the following theorem:

Theorem:

The two spaces $(R^n, d_2), (R^n, d_\infty)$ are not isometric for every dimension, $n \geq 2$.

Conclusions:

After having the result mentioned above proven, we can refer to the following issues in question:

1. What are the dimensions (n), assuming that such dimensions exist, by which the spaces $(R^n, d_1), (R^n, d_\infty)$ are isometric?

2. What are the dimensions (n), assuming that such dimensions exist, by which the spaces $(R^n, d_1), (R^n, d_2)$ are isometric?

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