Multi-Knowledge Electronic Comprehensive Journal For Education And Science Publications (MECSJ)
ISSUE (15), Dec (2018)

www.mescj.com

PROPERTIES OF AIRY FUNCTION AND APPLICATION TO THE V-SHAPE POTENTIAL

Salma Abdullah S Alshehri
Physics Department , Bisha University

Abstract

Airy function is a special function named after George Biddell Airy (1801–92). Airy function is used as a solution for Shcrodenger equation to study resonant tunneling in multilayered based on the exact solution of the Schrodinger equation under the application of a constant electric field, by using the transfer matrix approach, this paper concentrates and shows that the two functions of Airy have almost the same behavior as that of trigonometric functions for the negative values of x. But for positive values of x, Ai and its derivative are rapidly approaching zero, while Bi and its derivative tend rapidly to + infinity.

On the other hand, we define a_s and a'_s the sth zeros of Ai(x) and Ai'(x), b_s and b'_s the real zeros of Bi(x) and Bi'(x).

In this work, it is calculated energy eigenvalues for linear potential and solved the one-dimensional time-independent Schrodinger Without loss of generality we solved straightforwardly the case of Symmetric V-shape potential profile and the two turning points x1 and x2 which are defined by v(x)=E. Finally, the values of numerical solution approximately and the analytical solution approximately at a=1 show a very small error rate because we used approximation in the numerical solution and the analytical solutions are right for x >>1 .

Keywords: Airy function, Schrodinger equation.

1. Introduction

A historical background of Airy functions in physics, their derivatives and zeroes are presented in this paper. As application, we derive the odd and even solutions for the symmetric V-shape potential.

2. The Airy function in Physics.

The Airy function was introduced in 1838 by Sir George Biddell Airy (1801-1892), royal astronomer at Cambridge, when he was trying to analyze the intensity of light in the neighborhood of a caustic [1], which allows to explain diffraction in optics (and phenomena like rainbows).

They are mainly 3 equivalent definitions of the Airy function Ai(x): as a solution of the differential equation

$$y'' - xy = 0 \tag{1-1}$$

as an integral, or as a power series, namely

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xt+t^{2/3})}dt = \frac{1}{\pi 3^{2/3}}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{n+1}{3})}{n!}\sin\left(\frac{2(n+1)\pi}{3}\right)\left(3^{1/3}x\right)^{n} \tag{1-2}$$

Accordingly, there are many equivalent ways to write the Airy function in terms of special functions[2] , or Bessel functions. The notation Ai(x) is due to Miller in 1946 [3], who was in charge of the BAASMTC (British Association for Advancement of Science, Mathematical Tables Committee), a committee founded in 1871 and initially managed by Cayley, Rayleigh, Kelvin. This notation Ai(x) was quickly popularised by the book "Methods of Mathematical Physics" of Jeffreys & Jeffreys [4].

The Airy function Ai(x) oscillates on the real negative axis, where

$$\mathrm{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}x^{3/2} + \frac{1}{4}\pi\right)}{\sqrt{\pi}\,x^{1/4}} \tag{1-3}$$

And it has a discrete set of zeroes, while It decays exponential fast on the real positive axis, where

$$\mathrm{Ai}(x) \sim \frac{e^{-2/3}x^{3/2}}{2\sqrt{\pi}\,x^{1/4}} \tag{1-4}$$

The Airy function has many application in physics (optics, quantum mechanics, electromagnetic, radiative transfer) [5]. Why are there still nowadays so many articles involving this function? It is mainly because it has an intimate link with quantum mechanics, via the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right)\psi(x) = 0. \tag{1-5}$$

it implies that E has to be a zero of the Airy function! The equation is of the form

$$-a\,y'' + b\,x\,y = E\,y \tag{1-6}$$

and the physics of the Schrödinger equation implies that $y(\pm\infty) < \infty$, up to a change of variable, one recognizes the differential equation defining Ai(x), which in turn constraints E to belong to a discrete set of values

$$E = -a^{1/3}b^{2/3}\alpha_k \tag{1-7}$$

Multi-Knowledge Electronic Comprehensive Journal For Education And Science Publications (MECSJ)
ISSUE (15), Dec (2018)

www.mescj.com

(the α_k 's being the zeroes of Ai(x)). This quantization phenomena is thus typical in quantum mechanics (see [6]).

Other applications of the Airy function in physics are related to asymptotics expansions (Stokes phenomena, WKB method as initially investigated by Harold Jeffreys in 1923) [7].

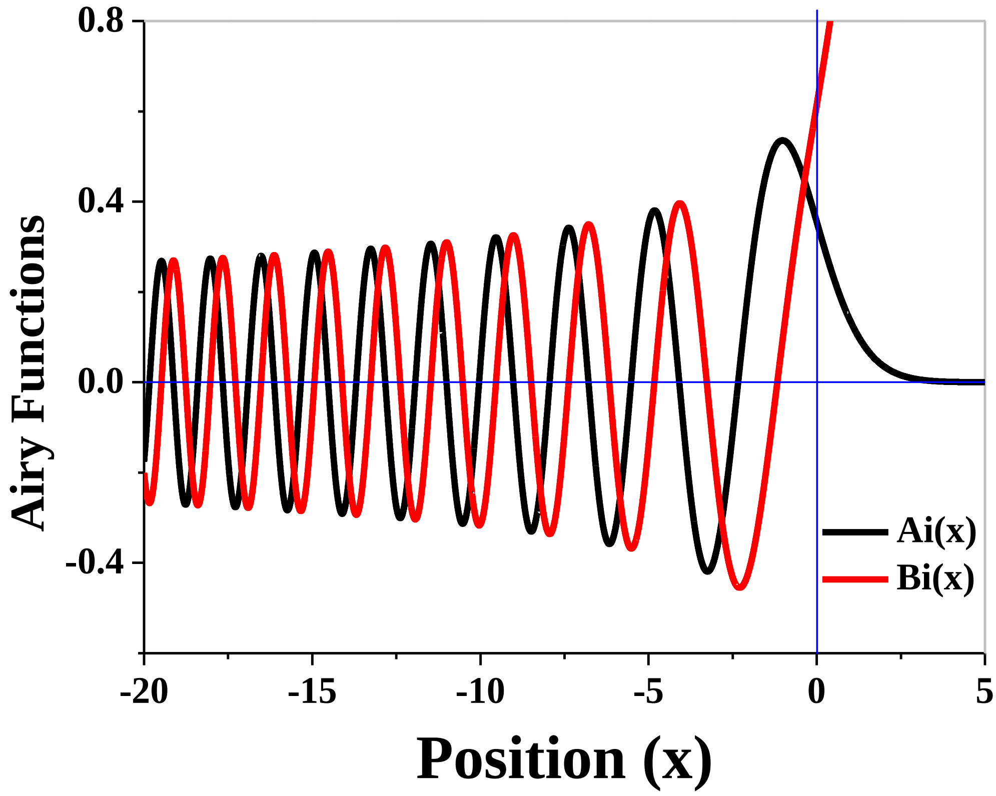

Fig.1.1. Variation of Airy functions as function of position (x).

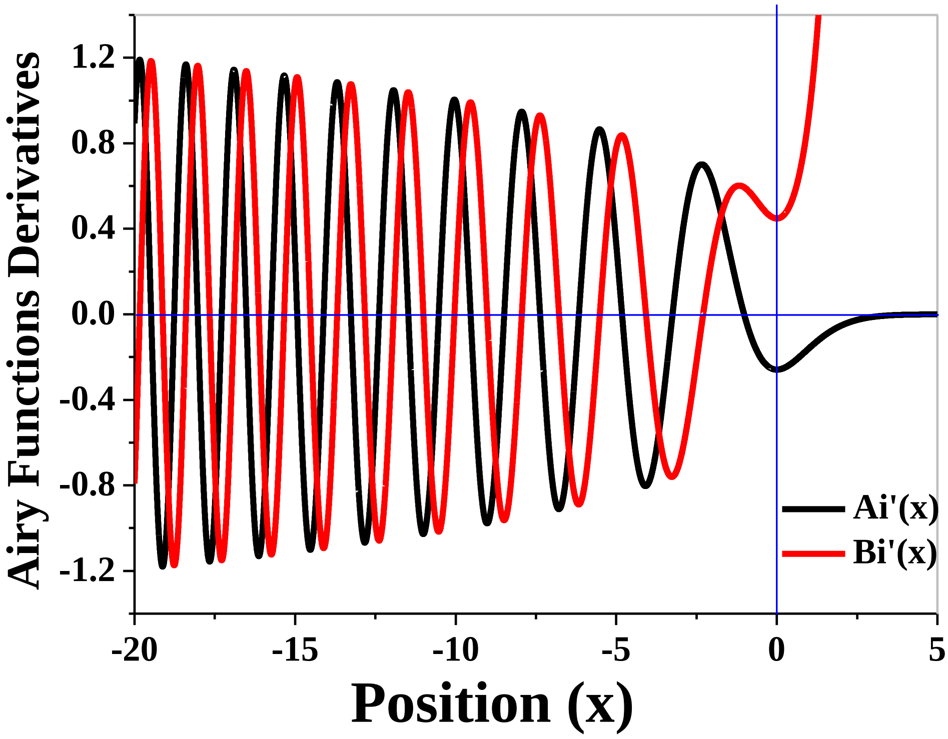

Fig.1.2. Variation of Airy functions derivetives as function of position (x).

In Fig.1.1 and Fig.1.2. It shows that the two functions have almost the same behavior as that of trigonometric functions for the negative values of x. But for positive values of x, Ai and its derivative are rapidly approaching zero, while Bi and its derivative tend rapidly to + infinity.

3. Zeros of Airy functions

Zeros of Airy function Ai(x) are located on the negative part of the real axis. Following the notation of Miller (1946), we define a_s and a'_s the s^{th} zeros of Ai(x) and Ai'(x), b_s and b'_s the real zeros of Bi(x) and Bi'(x) [8].

We thus obtain:

$$a_s = -f\left[\frac{3\pi}{8}(4s-1)\right] \tag{1-8}$$

$$a'_s = -g\left[\frac{3\pi}{8}(4s-3)\right] \tag{1-9}$$

The functions f(x) and g(x) are defined, with $|x| >> 1$, by the relations

$$f(x) \approx x^{2/3}\left(1 + \frac{5}{48x^2} - \frac{5}{36x^4} + \frac{77125}{82944x^6} - ....\right) \tag{1-10}$$

$$g(x) \approx x^{2/3}\left(1 - \frac{7}{48x^2} + \frac{35}{228x^4} - \frac{181223}{207360x^6} + ...\right) \tag{1-11}$$

4. The Symmetric V-shape potential

To calculate energy eigenvalues for linear potential, Fig.2.3. shows its profile. It is necessary to solve the one-dimensional time-independent Schrodinger Eq. (1-5). Without loss of generality, we assume throughout this paper, that $\hbar = 2m = 1$.

In region 1: $(x \le 0)$; $v(x) = -a x$

The Schrodinger equation in region 1, as express

12)-1(

$$\frac{d^2\psi_1}{dx^2} + (E + ax)\psi_1 = 0$$

$$\psi_1(x) = C_1 Ai(-a^{-2/3}(a x - E)) + D_1 Bi(-a^{-2/3}(a x - E)) \tag{1-13}$$

In region 2: $(x \ge 0)$; $v(x) = a x$

The Schrodinger equation in region 2, as express

$$\frac{d^2\psi_2}{dx^2} + (E - ax)\psi_2 = 0 \tag{1-14}$$

$$\psi_2(x) = C_2 Ai(a^{-2/3}(a x - E)) + D_2 Bi(a^{-2/3}(a x - E)) \tag{1-15}$$

Here, Ai and Bi are the Airy functions. For $x \to +\infty$, Bi and Bi' are divergent, then the constants D_1 and D_2 vanish. The continuity of these functions and their first derivative at $x = 0$, leads to:

$$Ai(-a^{-2/3}E)\,Ai(1, -a^{-2/3}E) = 0 \qquad (1\text{-}16)$$

Two solutions (even and odd) are possible for equation (1-16) [9].

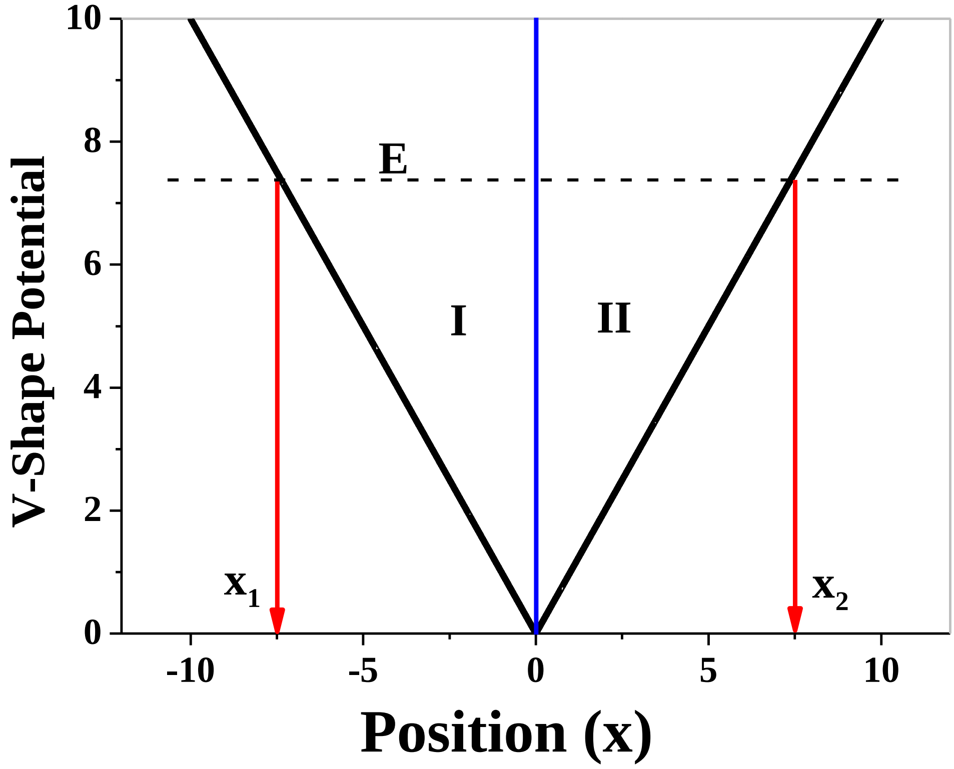

Fig1.3.V-shape potential profile and the two turning points x_1 and x_2 defined by v(x)=E.

a. <u>Even solutions</u>

Even solutions have an extremum at $x = 0$, and then the first derivative of the wave function must disappear at this point, thus

$$Ai'(-a^{-2/3}E) = Ai(1, -a^{-2/3}E) = 0 \qquad (1\text{-}17)$$

Using equation from (1-8) to (1-11) one can express the value energy as

$$a_n = -f\left[\frac{3\pi}{8}(4n-1)\right] = -f(\alpha_n) \qquad (1\text{-}18)$$

Where $\alpha_n = \frac{3\pi}{8}(4n-1)$

$$f(x) \approx x^{2/3}\left(1 + \frac{5}{48x^2} - \frac{5}{36x^4} + \frac{77125}{82944x^6} - ....\right) \qquad (1\text{-}19)$$

Then $-a^{-2/3}E_n = a_n = -f\left[\frac{3\pi}{8}(4n-1)\right]$

So $E_n = a^{2/3} f\left[\frac{3\pi}{8}(4n-1)\right]$

$$E_n = a^{2/3}.\left[\frac{3\pi}{8}(4n-1)\right]^{2/3}\left(1 + \frac{5}{48\alpha_n^{\ 2}} - \frac{5}{36\alpha_n^{\ 4}} + \frac{77125}{82944\alpha_n^{\ 6}} - ....\right)$$

(1-20)

This is the approximately analytical solution for even solutions.

b. Odd solutions

Odd solutions must pass by the origin, consequently null at $x = 0$, thus

$$Ai(-a^{-2/3}E) = 0 \qquad (1\text{-}21)$$

Using equation from (1-8) to (1-11) one can express the value of energy as

$$a'_n = -g\left[\frac{3\pi}{8}(4n-3)\right] = -g(\alpha'_n) \qquad (1\text{-}22)$$

where $\alpha'_n = \frac{3\pi}{8}(4n-3)$

$$g(x) \approx x^{2/3}\left(1 - \frac{7}{48x^2} + \frac{35}{228x^4} - \frac{181223}{207360x^6} + ...\right) \qquad (1\text{-}23)$$

Then $$-a^{-2/3}E_n=-g\left(\alpha_n'\right)$$

$$E_n=a^{2/3}g\left(\alpha_n'\right)$$

$$E_n=a^{2/3}.\left[\frac{3\pi}{8}\left(4n-3\right)\right]^{2/3}\left(1-\frac{7}{48\alpha_n'^2}+\frac{35}{228\alpha_n'^4}-\frac{181223}{207360\alpha_n'^6}+.......\right)$$

24)-1(

This is the approximately analytical for the odd solution, [8-9].

c. Approximately analytical and numerical solutions

Table 1.1. shows a comparison between the numerical solution approximately and the analytical solution approximately at a=1 and shows a very small error rate because we used approximation in the numerical solution and the the analytical solutions are right for x >>1 .

Numerical solution approximately	Analytical solution approximately	$\left\|\frac{\Delta E}{E}\right\|$
1.02	1.071395	0.049149
2.34	2.338641	0.000581
3.25	3.245766	0.001304
4.09	4.087952	0.000501
4.83	4.81654	0.002791
5.53	5.52056	0.001709
6.17	6.160181	0.001593

Table 1.1 .Comparison between the numerical solution approximately and the analytical solution approximately at a=1.

5. Conclusion

We have presented the historical backgrounds for the Airy functions in physics, described their behavior and derivatives. We noted the zeros of Airy function Ai(x) are located on the negative part of the real axis. Also, using Millers' notation for Airy functions zeroes, we solved straightforwardly the case of Symmetric V-shape potential and we have found the energy values numerically. The research of the analytical solutions it took considerable time and effort. We observed by comparing the numerical solution and the analytical solution , both results are found to be very close to each other's.

REFERENCES

[1] Airy, G. B. On the intensity of light in the neighbourhood of a caustic. Trans. Camb. Phil. Sec. 6 (1838).

[2] Jeffreys, H., and Jeffreys, B. S. Methods of Mathematical Physics. Cambridge, at the University Press, 1950. 2d ed. (1st ed. in 1946).

[3] Miller, J. C. P. The Airy Integral, Giving Tables of Solutions of the Differential Equation $y'' = xy$, vol. British Ass. Adv. Sc. Math. Tables Vol. B. Cambridge University Press, 1946.

[4] Nesvizhevsky, V. V. . A. Quantum states of neutrons in the earth’s gravitational field. Nature 415 (6869) (2002).

[5] Prudnikov, A. P., Brychkov, Y. A., and Marichev, O. I. Integrals and Series, 5 vols. Gordon and Breach, 1986–1992.

[6] Vallee, O., and Soares, M. Airy functions and applications to physics. Imperial College Press, London, 2010. 2nd ed. (1st English ed. in 2004, and 1st French ed. in 1998 by Dunod).

[7] Philippe Flajolet and the Airy Function by Cyril Banderier & Guy Louchard

[8] O. Vallee, M. Soares, Airy Functions and Applications to Physics, 2nd Ed., Imperial College Press, 2010 .

[9] F. Maiz, Physica B 449(2014)104 –108.